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A Galois Connection Approach to Superposition and Inaccessibility

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Working in a quantum logic framework and using the idea of Galois connections, we give a natural sufficient condition for superposition and inaccessibility to give the same closure map on sets of states.

1. INTRODUCTION

The ideas of superposition and disturbing measurement are central to quantum theory. In this paper we link these two ideas in a quantum logic framework making systematic use of Galois connections. After reviewing the elementary theory of Galois connections (Section 2), we consider two such, induced by two relations (Section 3).

First, we consider the relation between a state α and an event (yes–no proposition) a of making certain: $\alpha(a) = 1$. This relation induces a Galois connection, which itself induces the superposition closure map on the power set of states. Second, we consider the relation between states α, β of inaccessibility: that there is an event a such that $\alpha(a) = 1$ and $\beta(a) = 0$. This relation induces a Galois connection, and so a closure map on the power set of states.

It is then natural to ask for the conditions under which these closure maps coincide, especially since the idea of disturbing measurement links superposition and inaccessibility. Thus, suppose that with each event a there is associated a state-transition map Ga on the set of states, and that these Ga are of first kind in the usual sense. Then, as we shall see in Section 3, the ranges of the Ga must be closed under the superposition closure map,

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and the domains must be open in the associated interior map. Suppose, on the other hand, that inaccessibility constrains the Ga in that α is inaccessible to β iff for no a, $Ga(\beta) = \alpha$, i.e., iff β cannot transit to α under measurement of any event. Then, as we shall see, if inaccessibility is symmetric, we have that the ranges of the Ga must be closed under the inaccessibility closure map, and the domains must be open in the associated interior map.

In Section 4, we give a sufficient condition for these closure maps to coincide. The condition is natural in two senses: it is expressed in terms of corresponding closure maps on the power set of events, and it is equivalent to the conjunction of two assumptions familiar in quantum logic.

We will omit most proofs of results, since space is short and they are elementary. Besides, the connections between superposition and inaccessibility are familiar in quantum logic; so our aim in presenting these connections in terms of Galois connections is partly pedagogic. On the other hand, we think our results are of some technical interest: we stress that we will not require that the lattice L of events is orthomodular, nor that states be probability measures on L, nor that the set of states be σ-convex.

2. NOTATION AND BACKGROUND RESULTS

We first summarize the elementary theory of Galois connections; for clarity, we organize the material in subsections. We indicate a result by a "T" for "theorem". We will omit the proofs which are elementary and in some cases well known [e.g., some of what follows is in Abbott (1969, pp. 128–133)].

2.1. Closure Maps

A *closure map* on any poset $\langle X, \leq \rangle$ is a map $x \to x^*$ of X into itself s.t.: (C1) $x \leq y \Rightarrow x^* \leq y^*$ (isotone); (C2) $x \leq x^*$ (extensive); (C3) $x^* = x^{**}$ (idempotent). An element x of X is called *closed* iff $x = x^*$. An *interior map* on any poset $\langle X, \leq \rangle$ is a map $x \to x^\circ$ of X into itself s.t.: (C1); (C2°) $x^\circ \leq x$ (intensive); (C3). An element x of X is *open* iff $x = x^\circ$. A closure map * on an orthoposet (i.e., orthocomplemented poset) $\langle X, \leq, {}^\perp \rangle$ defines an associated interior map as follows: $x \to x^\circ := ([x^\perp]^*)^\perp$. And similarly, vice versa. So on an orthoposet, we can think of being given a closure map and interior map together. We shall concentrate on closure maps, saying little about interior maps.

T.2.1. If $\langle X, \leq \rangle = \langle X, \vee, \wedge \rangle$ is a complete lattice with a closure map *, then the closed elements of X form a complete lattice, X^* say, with the meet in X^* the same as the meet $\wedge$ in X, and the join in X^* equal to the closure of the join $\vee$ in X. (So x^* is the smallest closed element containing x.)

T.2.2. If $\langle X, \leq \rangle$ is a poset and $\langle Y, \leq \rangle$ is a complete sub-meet-semilattice of $\langle X, \leq \rangle$ s.t. for all $x \in X$ there is $y \in Y$ with $x \leq y$, then $\langle Y, \leq \rangle$ defines a closure map on X by $x^* = \bigwedge\{y \in Y: x \leq y\}$.

It follows that if $\langle X, \leq \rangle$ is a complete meet-semilattice with a 1, then the closure maps on X are in 1–1 correspondence with X's complete sub-meet-semilattices.

2.2. Relations Induce Closure Maps

Suppose given two sets X, Y, and a binary relation R from X to Y: we write xRy for $\langle x, y \rangle \in R$. R determines a converse relation $R^{\backprime}$ from Y to X. We define a map R from the power set $P(X)$ to $P(Y)$:

$$\text{if } x \in X, \quad R(x) := \{y \in Y: xRy\}$$

$$\text{if } W \subseteq X, \quad R(W) := \{y \in Y: \text{ for all } x \in W, xRy\}$$

As usual we interpret $R(\varnothing)$ vacuously, and set $R(\varnothing) = Y$ and $R^{\backprime}(\varnothing) = X$. R determines and is determined by the set $\{R(x): x \in X\}$.

T.2.3. The map $R: P(X) \to P(Y)$ is antitone; i.e., $U \subseteq W \Rightarrow R(W) \subseteq R(U)$. And for all $W \subseteq X$, $R(W) = \bigcap_{x \in W} R(x) = \bigcap_{U \subseteq W} R(U)$.

Therefore the family of sets $R(W)$, $W \subseteq X$, form a complete sub-meet-semilattice of $\langle P(Y), \subseteq \rangle$. Furthermore, Y, i.e., the 1 of $P(Y)$, is in this family since $R(\varnothing) = Y$. By T.2.2, this family defines a closure map on $P(Y)$: given any $Z \subseteq Y$, its closure Z^* is the smallest set of the form $R(W)$ containing it, for $W \subseteq X$.

Similarly the converse relation $R^{\backprime}$ gives a family $\{R^{\backprime}(Z): Z \subseteq Y\}$ of subsets of X; and this induces a closure map on $P(X)$.

2.3. Galois Connections

A *Galois connection* between a poset $\langle X, \leq \rangle$ and another poset $\langle Y, \leq \rangle$ is a pair of maps $\langle g, h \rangle$ s.t. both:

(i) $g: X \to Y$ and $h: Y \to X$ are each antitone; i.e., for $w, x \in X$, $w \leq x \Rightarrow g(w) \geq g(x)$; and similarly for h;

(ii) for all $x \in X$, $x \leq (h \circ g)(x) := h(g(x))$; and for all $y \in Y$, $y \leq (g \circ h)(y)$.

A classic source for the notion of a Galois connection is Ore (1944). The notion can be motivated in a way parallel to Blyth and Janowitz's motivation for the better-known notion of residuated map [one takes antitone analogs of Theorems 2.1–2.7 of Blyth and Janowitz (1971).] But for us it is more important to describe how Galois connections are related to closure maps and relations.

2.4. A Galois Connection Induces a Closure Map

First we have:

T.2.4. If $\langle g, g' \rangle$ is a Galois connection between posets X and Y, then the composition $(g' \circ g)$ is a closure map on X and $(g \circ g')$ is a closure map on Y. The closed elements in X (resp. Y) are the images of g' (resp. g). Any element in X (resp. Y) has the same image under g (resp. g') as its closure: that is,

$$g(x) = (g \circ g' \circ g)(x); \qquad g'(y) = (g' \circ g \circ g')(y)$$

We can also extend T.2.1, using the idea of dual isomorphism. Recall that isomorphism for posets is not just a matter of there being an isotone bijection from one to the other. For an isotone bijection might not have an isotone inverse. So we say that two posets are *isomorphic* iff there is an isotone bijection between them, f say, with an isotone inverse. This is equivalent to: $x \leq y$ iff $f(x) \leq f(y)$. Similarly, an antitone bijection might not have an antitone inverse. So we way that two posets are *dually isomorphic* iff there is an antitone bijection between them, f say, with an antitone inverse. This is equivalent to $x \leq y$ iff $f(x) \geq f(y)$. Then we can extend T.2.1 as follows:

T.2.5. If $\langle g, h \rangle$ is a Galois connection between complete lattices X and Y, then the sets $X(h \circ g)$ and $Y(g \circ h)$ of closed elements of X and Y, respectively, are complete lattices and g and h are dual isomorphisms between them. That is, g and h are antitone bijections on the sets of closed elements, and the products $(h \circ g)$ and $(g \circ h)$ are the identity maps on $X(h \circ g)$ and $Y(g \circ h)$, respectively.

2.5. A Relation Induces a Galois Connection

T.2.6. A relation R between X and Y induces a Galois connection between $P(X)$ and $P(Y)$; via the maps in Section 2.2, $R: P(X) \to P(Y)$ and $R^{\backprime}: P(Y) \to P(X)$.

Remarks. (1) By T.2.4, a closure map $Z \to R(R^{\backprime}(Z))$ [resp. $W \to R^{\backprime}(R(W))$] is defined on $P(Y)$ [resp. $P(X)$]. And for $W \subseteq X$, $Z \subseteq Y$: $R(W) = R(R^{\backprime}(R(W)))$ and $R^{\backprime}(Z) = R^{\backprime}(R(R^{\backprime}(Z)))$. (2) Since $P(X)$ and $P(Y)$ are complete lattices, T.2.5 implies that the lattices of closed elements are dual isomorphic, the dual isomorphisms taking meets into joins and vice versa. In Section 4 we shall use the formula for the reversal of joins and meets on subsets $Z_1, Z_2 \subseteq Y$. So we state it separately:

Corollary to T.2.6. If R is a relation from X to Y, and $Z_1, Z_2 \subseteq Y$, then:

$$R(R`(Z_1) \cap R`(Z_2)) = R(R`(Z_1 \cup Z_2))$$

$$R(R`(Z_1) \cup R`(Z_2)) = R(R`(Z_1)) \cap R(R`(Z_2))$$

(3) The closure map induced by a relation via the Galois connection it induces is the same as the closure map in Section 2.2, as in the following:

T.2.7. If R is a relation from X to Y, and $Z \subseteq Y$, then the smallest set Z^*, of the form $R(W)$, $W \subseteq X$, containing Z is $R(R`(Z))$; and similarly for $W \subseteq X$.

It will be convenient to make the following definition. If R is a relation from X to Y, and $W \subseteq X$ with $W = R`(R(W))$, I will say that $W$ is *R-closed.* Similarly, $Z \subseteq Y$ with $Z = R(R`(Z))$ is *R`-closed.*

3. THE SUPERPOSITION AND INACCESSIBILITY LATTICES

We first review how the existence of superpositions endows the power set of states with a closure map, which gives a complete lattice of closed sets of states. We remark that the domains and ranges of the state-transition maps for first-kind measurements are respectively open and closed sets. We then assume that states also have a binary relation I of inaccessibility that constrains these domains and ranges. In accordance with T.2.3 (or T.2.4 and T.2.6), I induces a closure map, and so a complete lattice. We thus have two closure maps and two lattices which are interpreted in terms of first-kind measurement.

Notation: We use Σ for the set of all states; α, β, γ, for states; T, U, for sets of states; L for the set of events; a, b, c for events; A, B, C for sets of events. We will introduce in a piecemeal fashion the assumptions ("A") about events and states that we need, so as to clarify which results depend on which assumptions. As mentioned in the Introduction, we shall not need to assume that the lattice of events is orthomodular, nor that the states are probability measures, nor that the set of states is σ-convex.

3.1. The Superposition Lattice $\Sigma^\wedge$

To define this we only need assume:

A.3.0. L is a nonempty set, and Σ a nonempty set of functions from L to $[0, 1]$. For any $A \subseteq L$, and for any $T \subseteq \Sigma$, we define

$$\mathrm{S1}(A) := \{\alpha \in \Sigma\colon \text{for all } a \in A,\ \alpha(a) = 1\}$$

$$\mathrm{L1}(T) := \{a \in L\colon \text{for all } \alpha \in T,\ \alpha(a) = 1\}$$

[Here, it is to be understood that $S1(\emptyset) = \Sigma$, and $L1(\emptyset) = L$.] We then define

$$T^\wedge := S1(L1(T))$$

$\alpha \in T^\wedge$ is called a superposition of elements of T; $^\wedge$ is the superposition map.

Superposition is a closure map on the power set $P(\Sigma)$. Since $P(\Sigma)$ is a complete lattice under set-inclusion, T.2.1 implies that the set of $^\wedge$-closed sets of states forms a complete lattice with set-intersection as meet, and closure of set union as join. We call this lattice $\Sigma^\wedge$.

The closure map $^\wedge$ is induced by a Galois connection which is itself induced by a relation. For observe that (L1, S1) is a Galois connection between $P(\Sigma)$ and $P(L)$; and the closure map it induces in $P(\Sigma)$ is the superposition operation. T.2.5 then implies that $\Sigma^\wedge$ is dual isomorphic by the map L1 to the sets of events that are closed under the closure map on $P(L)$ given by $A \to L1(S1(A))$. Furthermore, the relation from Σ to L, "α makes a certain," i.e., $\alpha(a) = 1$, induces this Galois connection by T.2.6.

We can also define:

$$S0(A) := \{\alpha \in \Sigma\text{: for all } a \in A, \alpha(a) = 0\}$$

$$L0(T) := \{a \in L\text{: for all } \alpha \in T, \alpha(a) = 0\}$$

and show similarly that $T \to S0(L0(T))$ is a closure map on $P(\Sigma)$ induced by the relation from Σ to L, "α makes a impossible," i.e., $\alpha(a) = 0$. However, given a natural assumption, this map is the same as $T \to S1(L1(T))$. The assumption which we make from now on is:

A.3.1. L is an orthoposet; the states in Σ "mesh" with L's orthocomplement $^\perp$, in the sense that, for all α, for all a: $\alpha(a) = 0$ iff $\alpha(a^\perp) = 1$.

(Think of $^\perp$ as transposing the Yes and No labels for events in L.) We then have:

T.3.1. For all $T \subseteq \Sigma$, $S1(L1(T)) = S0(L0(T))$.

3.2. The Superposition Lattice $\Sigma^\wedge$ and Measurement

Now suppose that measurements disturb states, in the sense that for each event a, there is a map Ga on the states that gives the transitions which states undergo when yielding Yes for measurement of a. And suppose that the Ga are of *first kind*, in the sense that:

(i) If $\alpha(a) = 1$, then $Ga(\alpha) = \alpha$; i.e., Ga fixes states that make a certain.

(ii) For all α, $[Ga(\alpha)](a) = 1$; i.e., any state arising from a Yes outcome makes a certain.

It follows from this that the ranges of the maps Ga are closed under superposition. For (i) implies that $S1(a) \subseteq \text{ran}(Ga)$; and (ii) implies that

$\mathrm{ran}(Ga) \subseteq \mathrm{S1}(a)$. So $\mathrm{S1}(a) = \mathrm{ran}(Ga)$. But $\mathrm{S1}(a)$ must be $^\wedge$-closed: by T.2.4, or directly by observing that if $\alpha \in \mathrm{S1}(\mathrm{L1}(\mathrm{S1}(a)))$, i.e., α makes certain all the events that all of $\mathrm{S1}(a)$ agree as certain, then α must make a certain. [Section 4 will treat the converse: is every $^\wedge$-closed set a $\mathrm{ran}(Ga)$ for some Ga?]

The relation of $\Sigma^\wedge$ to the $\mathrm{dom}(Ga)$ is a bit more subtle, given that we do not assume that the states are measures. The domains are usually defined by one of

$$\mathrm{dom}(Ga) := \{\alpha : \alpha(a^\perp) \neq 1\} \qquad \text{or} \qquad \mathrm{dom}(Ga) := \{\alpha : \alpha(a) \neq 0\}$$

These are trivially equivalent if the states are measures; and equivalent iff A.3.1. And once we recall that $\mathrm{ran}(Ga) = \mathrm{S1}(a)$, A.3.1 is plainly equivalent to

$$(\mathrm{S1}(a^\perp))^c = (\mathrm{ran}(Ga^\perp))^c = \mathrm{dom}(Ga)$$

Recalling from Section 2.1 that any closure map $^\wedge$ on a power set has an associated interior map: $T \to T^\circ := ((T^c)^\wedge)^c$, it follows that $\mathrm{dom}(Ga)$ is open in the associated interior map for superposition [take $T = (\mathrm{S1}(a^\perp))^c$].

3.3. The Inaccessibility Lattice $\Sigma\#$

Now suppose that there is a binary relation Q on Σ that constrains the maps Ga in the sense that $\alpha Q \beta := \alpha$ is accessible to β iff $\beta Q^\backprime \alpha := \beta$ can transit to α under measurement of some event a. Q constrains the Ga's in that Q is given once and for all on Σ: there is not a separate relation for each a. But for now, we need not choose a particular Q (we will do so in Section 4).

Recall from Sections 2.2, 2.4, and 2.5 that any relation R on Σ induces a complete lattice of R-closed sets of states, i.e., sets T s.t. $R^\backprime(R(T)) = T$. So one could consider the Q-closed sets forming an accessibility lattice. It turns out that the $\mathrm{ran}(Ga)$ and $\mathrm{dom}(Ga)$ (and so by Section 3.2, $\Sigma^\wedge$) are connected not to accessibility, but to its complement relation, inaccessibility, I. And this connection requires that inaccessibility is symmetric; but it does not need A.3.1. To be precise, we have:

T.3.2. If inaccessibility I is symmetric, every $\mathrm{ran}(Ga)$ is I-closed.

Proof. Consider $\mathrm{ran}(Ga)$, for Ga of first kind, in terms of accessibility, or its converse, "transitability." We get the condition $\alpha \in \mathrm{ran}(Ga)$ iff α cannot transit to any state that itself cannot transit to any element of $\mathrm{ran}(Ga)$. If Q (equivalently, its complement I) is symmetric, then this condition is equivalent to $\mathrm{ran}(Ga)$ being I-closed. Using Section 2.6, T.2.10, and definition following, we have $T \subseteq \Sigma$ is I-closed iff $T = I^\backprime(I(T))$ iff any α that bears I to all of $I(T)$ is in T iff any nonmember of T does not bear

I to all of $I(T)$ iff for any nonmember γ of T there is a $\beta \in I(T)$ s.t. β can transit to γ. Provided I is symmetric, this condition for T being I-closed is the above condition on ran(Ga). ■

Notation. We recall the complete lattice of I-closed sets of states the *inaccessibility lattice*. We write the closure map $(I \circ I)$ on $P(\Sigma)$ as $\#$, and we call the inaccessibility lattice $\Sigma\#$.

Remark. The definition of $\Sigma\#$ by the map I on $P(\Sigma)$ generalizes a familiar example: the definition of the lattice of subspaces of a vector space that is equipped with an inner product by the annihilators of sets of vectors.

Turning to domains, it is easier to prove dom(Ga) is I-open (use T.2.6, Remark 1).

T.3.3. If I is symmetric, every dom(Ga) is I-open.

Remark. If we add assumption A.3.1, we can avoid the complicated proof of T.3.2 because T.3.2 becomes a corollary of T.3.3: given dom(Ga) is I-open, we conclude by A.3.1 that ran$(Ga^{\perp}) = (\text{dom}(Ga))^c$ is I-closed.

Finally, note that accessibility being symmetric and reflexive makes its complement I symmetric and irreflexive. Then the map I: $P(\Sigma) \to P(\Sigma)$ is an orthocomplement on the I-closed sets. For we have (using T.2.5):

T.3.4. If R is a symmetric and irreflexive relation on Σ, then the map R: $P(\Sigma) \to P(\Sigma)$ is an orthocomplement on the complete lattice of R-closed subsets of Σ.

Furthermore, it is natural to assume that I is irreflexive; i.e., Q is reflexive, every state can transit to itself under measurement, every state makes some event certain. This holds if the trivial event 1 (whose existence is secured by L's being an orthoposet, A.3.1) satisfies: $\alpha(1) = 1$ for all $\alpha \in \Sigma$. In fact we shall need to assume this later on and so we adopt:

A.3.2. For all $\alpha \in \Sigma$, $\alpha(1) = 1$.

In view of A.3.1, an equivalent statement is: for all $\alpha \in \Sigma$, $\alpha(0) = 0$.

4. THE IDENTITY OF THE LATTICES $\Sigma^\wedge$ AND $\Sigma\#$

Our main aim is to give a sufficient condition, cast in terms of Galois connections, for the lattices $\Sigma^\wedge$ and $\Sigma\#$ to be identical. Section 4.1 chooses an inaccessibility relation. Section 4.2 presents the condition, arguing that it is natural. Section 4.3 proves that the condition is sufficient. Section 4.4 shows that the condition is equivalent to the conjunction of two assumptions familiar in quantum logic. Section 4.5 gives counterexamples to some conjectures that arise. Section 4.6 relates our results to some previous work.

4.1. Choosing the Inaccessibility Relation

The lattice $\Sigma\#$ of course varies with the choice of the relation I on Σ. Recall that in Hilbert space quantum theory, an ideal measurement projects rays (states) into the $(a = 1)$-eigenspace; so inaccessibility is orthogonality. This suggests the definition

$$\alpha I \beta \quad \text{iff} \quad \text{there is } a \in L \text{ with } \alpha(a) = 0,\ \beta(a) = 1$$

So defined, I is irreflexive. And A.3.1 implies that it is symmetric. From now on, we shall assume this choice of the relation I. We will need some facts about the relation between I and the orthocomplement $^{\perp}$ in L. A.3.1 gives us:

T.4.1. $\alpha I \beta$ iff there is $a \in L$ s.t. $\alpha(a) = 1$, $\beta(a^{\perp}) = 1$
iff there is $a \in L$ s.t. $\alpha(a) = 0$, $\beta(a^{\perp}) = 0$

We will later need to assume that the states are isotone functions on L, and this enables us immediately to relate I to orthogonality in L. So now assume:

A.4.1. States are isotone: for all events and states, $a \leq b \Rightarrow \alpha(a) \leq \alpha(b)$.

T.4.2. $\alpha I \beta$ iff there are $a, b \in L$ s.t. a is orthogonal to b, $\alpha(a) = 1$, $\beta(b) = 1$.

4.2. Motivating the Sufficient Condition of Identity

For the lattices $\Sigma^\wedge$ and $\Sigma\#$ to be identical, the ranges of the two closure maps $^\wedge$ and $\#$ must be the same, for the ranges just are the lattices of closed elements. In fact, the maps must be identical, as shown by:

T.4.3. $\Sigma^\wedge = \Sigma\#$ iff for all $T \subseteq \Sigma$, $T^\wedge = T\#$.

We therefore ask: under what conditions does a closure map on $P(\Sigma)$ induced via a Galois connection by a relation I on Σ coincide with a closure map on $P(\Sigma)$ induced by a Galois connection $\langle \text{L1}, \text{S1} \rangle$ (or equivalently, in view of T.3.1: by $\langle \text{L0}, \text{S0} \rangle$) from $P(\Sigma)$ to $P(L)$?

To tackle this question, we first remark that an analogous question can be asked about $P(L)$, but, as we shall see, the analogous question is not equivalent. For, recall that (i) orthogonality, being a relation on L, induces a Galois connection and so a closure map on $P(L)$; (ii) the Galois connection $\langle \text{S1}, \text{L1} \rangle$ from $P(L)$ to $P(\Sigma)$ induces a closure map on $P(L)$: $A \to \text{L1}(\text{S1}(A))$; (iii) similarly to (ii), the Galois connection $\langle \text{S0}, \text{L0} \rangle$ induces a closure map on $P(L)$. So we can ask: do the closure maps on $P(L)$ induced by $\langle \text{S1}, \text{L1} \rangle$ and $\langle \text{S0}, \text{L0} \rangle$ coincide, like the corresponding maps on $P(\Sigma)$? And whether or not they coincide, under what conditions does the closure map on $P(L)$ induced by orthogonality coincide with either of them?

Furthermore, one naturally expects these questions about $P(L)$ to be related to those about closure maps on $P(\Sigma)$, since I on Σ is related to orthogonality in L (T.4.1, T.4.2). So one naturally conjectures: the same closure map on $P(\Sigma)$ is induced by inaccessibility and by superposition iff the same closure map on $P(L)$ is induced by orthogonality and by either $A \to \mathrm{L1}(\mathrm{S1}(A))$ or $A \to \mathrm{L0}(\mathrm{S0}(A))$.

As we shall see, the "if" statement is true when we make an appropriate choice, namely $A \to \mathrm{L0}(\mathrm{S0}(A))$, for the closure map on $P(L)$ corresponding to superposition. In other words, we have a sufficient condition, natural from the point of view of Galois connections, for the identity of $\Sigma^\wedge$ and $\Sigma\#$. But the "only if" statement is false (the reason for this asymmetry is essentially that L is a poset, while Σ is not). We spend the rest of this subsection describing the choice $A \to \mathrm{L0}(\mathrm{S0}(A))$.

Note first that the closure maps on $P(L)$ induced by $\langle \mathrm{S1}, \mathrm{L1} \rangle$ and $\langle \mathrm{S0}, \mathrm{L0} \rangle$ do not coincide. By A.4.1, $\mathrm{L1}(\mathrm{S1}(A))$ is a filter in L and $\mathrm{L0}(\mathrm{S0}(A))$ is an ideal in L. We can say more, if we now assume:

A.4.2. L is complete.

With this assumption, we get this result, suggesting the choice $A \to \mathrm{L0}(\mathrm{S0}(A))$.

T.4.4. The closure map on $P(L)$ induced by orthogonality $\perp$ is $A \to \perp\perp(A) = (\sup A)\downarrow$. So the $\perp$-closed sets are principal ideals in L.

Corollary to T.4.4. The closure maps on $P(L)$ induced by orthogonality $\perp$ and by $\langle \mathrm{S0}, \mathrm{L0} \rangle$ coincide iff for all $A \subseteq L$, $(\sup A)\downarrow = \mathrm{L0}(\mathrm{S0}(A))$; iff for all $A \subseteq L$, $\sup[\mathrm{L0}(\mathrm{S0}(A))] = \sup A \in \mathrm{L0}(\mathrm{S0}(A))$.

Note, however, that we could work with $A \to \mathrm{L1}(\mathrm{S1}(A))$ by "reversing the order in the lattice L"; i.e., by using the relation $\perp'$ say, defined in terms of $\perp$ by

$$a \perp' b \text{ iff } a^\perp \perp b^\perp; \quad \text{i.e., iff } a \geq b^\perp$$

This gives us (again using A.4.2): The closure map on $P(L)$ induced by $\perp'$ is $A \to \perp'\perp'(A) = (\inf A)\uparrow$. So the closure maps induced by $\perp'$ and by $\langle \mathrm{S1}, \mathrm{L1} \rangle$ coincide iff for all $A \subseteq L$, $(\inf A)\uparrow = \mathrm{L1}(\mathrm{S1}(A))$; iff $\inf[\mathrm{L1}(\mathrm{S1}(A))] = \inf A \in \mathrm{L1}(\mathrm{S1}(A))$.

Furthermore, one pair of maps coincides iff the other does. That is: $\perp\perp = \mathrm{L0}.\mathrm{S0}$ iff $\perp'\perp' = \mathrm{L1}.\mathrm{S1}$. To prove this, we define: for all $A \subseteq L$, $A^\perp := \{a \in L : a^\perp \in A\}$. It is then easy to show: $b \in A\uparrow$ iff $b^\perp \in (A^\perp)\downarrow$; and so, $(\sup A)^\perp = \inf(A^\perp)$. And then it is easy to prove:

T.4.5. $\sup[\mathrm{L0}(\mathrm{S0}(A))] = \sup A \in \mathrm{L0}(\mathrm{S0}(A))$ iff $\inf[\mathrm{L1}(\mathrm{S1}(A))] = \inf A \in \mathrm{L1}(\mathrm{S1}(A))$.

Thus we can choose whether to work either with $\perp$ and $\langle \mathrm{S0}, \mathrm{L0} \rangle$, or with $\perp'$ and $\langle \mathrm{S1}, \mathrm{L1} \rangle$. We choose the former.

4.3. Sufficiency

To prove the superposition and inaccessibility lattices identical from the assumption $\perp\perp(A) = \mathrm{L0}(\mathrm{S0}(A))$, we need the notion, familiar in quantum logic, of *supports* (carriers). Supports are normally defined while assuming that states are probability measures on L, assumed to be an orthomodular poset. Thus one normally defines

$$a \text{ is the support of } \alpha \text{ iff } \mathrm{L1}(\alpha) = \{b \in L\colon a \leq b\}$$

The support if it exists is unique and is written $s(\alpha)$. The fact that α is a probability measure implies another useful characterization:

$$a = s(\alpha) \text{ iff for all } b, \alpha(b) = 0 \text{ iff } a \perp b$$

Thus $a = s(\alpha)$ iff $\mathrm{L0}(\alpha) = \perp(a)$; and $\mathrm{L0}(\alpha) = [s(\alpha)^{\perp}]\downarrow$. Similarly, one defines for a set of states T:

$$a \text{ is } s(T) \quad \text{iff } \mathrm{L1}(T) = \{b \in L\colon a \leq b\}; \quad \text{or} \quad \text{iff } \mathrm{L0}(T) = \{b \in L\colon a \perp b\}$$

so that $\mathrm{L0}(T) = (s(T)^{\perp})\downarrow$.

However, states need not be probability measures for the definition of supports, nor for the above characterizations. It is easy to check that A.3.1, A.3.2, and A.4.1 together imply the equivalence of the above characterizations of support. And A.4.2 (L is complete) gives two further usual results, without states being measures:

(1) If each state has a support, then the support of any set of states must exist and it is the join of its members' supports:

(2) α has a support iff $\forall A \subseteq L$, if $\forall a \in A$, $\alpha(a) = 1$, then $\alpha(\inf A) = 1$.

That is, α has a support iff $\forall A \subseteq L$, if $\forall a \in A$, $\alpha(a) = 0$, then $\alpha(\sup A) = 0$.

We now have two lemmas and then the main result:

T.4.6. $\perp\perp(A) = \mathrm{L0}(\mathrm{S0}(A))$ implies that every set of states has a support.

T.4.7. $\perp\perp(A) = \mathrm{L0}(\mathrm{S0}(A))$ implies that for all $T \subseteq \Sigma$, $I(T) = \mathrm{S0}(s(T))$.

Proof. We prove this by proving

$$I(T) = \mathrm{S0}(U_{\alpha \in T}\{s(\alpha)\}) = \mathrm{S0}(s(T))$$

For the first equation, $\mathrm{S0}(U_{\alpha \in T}\{s(\alpha)\}) \subseteq I(T)$ is clear from Section 4.1's choice of the inaccessibility relation. A.4.1, isotonicity of states, yields the converse. For the second equation, use the fact that every set of states has a support, $s(T) = \bigvee_{\alpha \in T} s(\alpha)$, as follows. The $\supseteq$ is trivial by A.4.1. For the converse, $\subseteq$, suppose $\beta \in$lhs: for all $\alpha \in T$, $\beta \in \mathrm{S0}(s(\alpha))$. So $\beta \in \mathrm{S1}(s(\alpha)^{\perp})$; so, by de Morgan's laws, we have $s(\beta) \leq \bigwedge_{\alpha \in T} s(\alpha)^{\perp} = (s(T))^{\perp}$. So $\beta(s(T)) = 0$.

■

Finally we get:

T.4.8. Suppose that for all A, $\perp\perp(A) = \mathrm{L0}(\mathrm{S0}(A))$. Then for all T, $\mathrm{S0}(\perp(\mathrm{L0}(T))) = I(T)$, and so $II(T) = \mathrm{S0}(\mathrm{L0}(T))$.

Proof. $\mathrm{S0}.\perp.\mathrm{L0}(T) = \mathrm{S0}.\perp.[(s(T)^{\perp})\downarrow] = \mathrm{S0}.\perp.\perp.(s(T)) = \mathrm{S0}.\mathrm{L0}.\mathrm{S0}(s(T)) = \mathrm{S0}(s(T)) = I(T)$. (The last two equations hold because of T.2.4, and T.4.7, respectively.) This implies $II(T) = \{\mathrm{S0}.\mathrm{L0}\}(T)$ [twice using Section 2.1]. ■

4.4. The Content of $\perp\perp = \mathrm{L0S0}$

Given our assumptions (especially that L is complete), it turns out that our sufficient condition for $\Sigma^\wedge = \Sigma\#$ is equivalent to the conjunction of two conditions familiar in quantum logic: the existence of supports, in Section 4.3's sense; and the set of states being strongly ordering in the sense of Beltrametti and Cassinelli (1981, p. 116), i.e.:

Strong ordering: For all a, b, if $\mathrm{S1}(a) \subseteq \mathrm{S1}(b)$, then $a \leq b$. [Such a set of states is also called "rich" (Pták and Pulmannová (1991, p. 21).] As we saw in Section 4.3 about supports, this condition is normally considered for states as measures. But we consider it on our more general states. In particular, our A.3.1 is enough to get the equivalence, useful for us:

T.4.9. The states are strongly ordering iff if $\mathrm{S0}(a) \subseteq \mathrm{S0}(b)$, then $a \geq b$.

We have already seen that $\perp\perp = \mathrm{L0S0}$ implies that every set of states has a support (T.4.6). We also have (using T.4.4):

T.4.10. $\perp\perp = \mathrm{L0S0}$ implies strong ordering.

Conversely, we have, using L complete, isotonicity, and T.4.9 (so A.3.1):

T.4.11. If every state has a support and the states are strongly ordering, then $\perp\perp = \mathrm{L0S0}$.

Proof. The existence of supports and the second equation of T.4.7 imply: if $\alpha \in \mathrm{S0}(A)$, then $\alpha(\sup A) = 0$. So $(\sup A)\downarrow \subseteq \mathrm{L0S0}(A)$. And $(\sup A)\downarrow = \perp\perp(A)$ by T.4.4. Now we show the converse inclusion $\supseteq$ using strong ordering. Suppose $b \in \mathrm{L0S0}(A)$, so $\mathrm{S0}(A) \subseteq \mathrm{S0}(b)$. Supports and isotonicity imply that $\mathrm{S0}(\sup A) = \mathrm{S0}(A)$. So $\mathrm{S0}(\sup A) \subseteq \mathrm{S0}(b)$. Strong ordering implies that $b \leq \sup A$. So $\mathrm{L0S0}(A) \subseteq (\sup A)\downarrow = \perp\perp(A)$. ■

4.5. Examples

First, we give a counterexample to Section 4.3's condition being necessary. Consider the lattice and the set of just two states shown in Fig. 1, where;

States	a	b	c	$a^{\perp}$	$b^{\perp}$	$c^{\perp}$
α	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
β	1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$

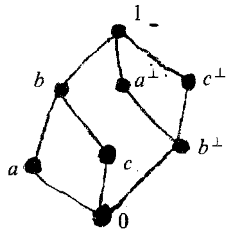

Fig. 1

Then we have $\alpha I\beta$, $II(\alpha) = \{\alpha\}$, $II(\beta) = \{\beta\}$, and $II = \text{S0.L0}$. But we also have

$$\text{L0.S0}(\{\beta^\perp\}) = \text{L0}(\beta) = \{a^\perp, b^\perp, 0\} \neq \{b^\perp, 0\} = b^\perp\downarrow = \perp\perp(\{a, c\})$$

and strong ordering fails: $a^\perp \neq b^\perp$ while $\text{S0}(a^\perp) = \text{S0}(b^\perp) = \{\beta\}$.

Since our condition $\perp\perp = \text{L0S0}$ is sufficient but not necessary for $II = \text{S0L0}$, one asks whether it can be weakened while remaining sufficient. In view of Section 4.4, the obvious question is: can one of the conjuncts, supports or strong ordering, be dropped?

The answer is No. Agreed, we can drop strong ordering and still prove "half" of $II = \text{S0L0}$. For we have, using A.3.1, A.3.2, A.4.1 and A.4.2 (i.e., L is a complete orthocomplemented lattice, and states are isotone functions that "mesh" with L's orthocomplement):

T.4.12. If every state has a support, then for all $T \subseteq \Sigma$: $\text{S0L0}(T) \subseteq II(T)$.

Proof. Pick any T, $\alpha \in \text{S0L0}(T)$, and $\beta \in I(T)$. We need to show $\alpha I\beta$, i.e., to find an event a s.t. $\alpha(a) = 0$ and $\beta(a) = 1$. We construct a from those events on which β disagrees with elements of T. Define $a := \bigwedge_{\gamma\in T} \{b \in L\colon \beta(b) = 1 \text{ and } \gamma(b) = 0\}$. By isotonicity, $a \in \text{L0}(T)$, so $\alpha(a) = 0$. By supports [cf. (2) just before T.4.6], $\beta(a) = 1$. ■

But we cannot strengthen the consequent of T.4.12 to $\text{S0L0}(T) = II(T)$. For consider the lattice above, but now add to α and β a third state:

	a	b	c	$a^\perp$	$b^\perp$	$c^\perp$
γ	0	0	0	1	1	1

Then $\alpha I\beta I\gamma$, not $(\alpha I\gamma)$, and the conditions of T.4.12 are satisfied; but

$$\text{S0L0}(\gamma) = \text{S0}(\{a, b, c\}) = \{\gamma\} \neq \{\alpha, \gamma\} = II(\gamma)$$

Again, strong ordering fails: $\text{S1}(c^\perp) \subseteq \text{S1}(a^\perp)$.

On the other hand, if we drop supports but retain strong ordering, we cannot prove the converse inclusion of T.4.12: we cannot prove

$II(T) \subseteq$ S0L0(T). For consider the following counterexample using six events and eight states (the smallest we have found):

States	a	b	c	$a^\perp$	$b^\perp$	$c^\perp$
α	0	1	0	1	0	1
β	1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
γ	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1
δ	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0
ε	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
ζ	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1
η	$\frac{1}{2}$	0	1	$\frac{1}{2}$	1	0
φ	1	0	$\frac{1}{2}$	0	1	$\frac{1}{2}$

This gives $I(\alpha) = \{\beta, \delta, \eta, \varphi\}$, so that $\gamma \in II(\alpha)$. But γ is not a member of S0L0$(\alpha) = \{\alpha\}$. And S1$(a) = \{\beta, \varphi\}$, S1$(b) = \{a, \beta\}, \ldots,$ S1$(c^\perp) = \{\alpha, \gamma, \zeta\}$; so that for no events x and y do we have S1$(x) \subseteq$ S1(y). So strong ordering holds vacuously, and we can take the lattice to be as given in Fig. 2.

4.6. Relation to Previous Work

We end by briefly relating our results to some previous work. As we said in Section 1, the novelties here are the explicit use of Galois connections and the allowance of nonorthomodular lattices and of states that are not probability measures. We shall follow the order in which we introduced the main concepts: considering superposition, then inaccessibility, and then very briefly supports.

Our superposition closure map $T \to T^\wedge$ is the usual notion of superposition within quantum logic. It goes back at least to Varadarajan (1968, pp. 116–117, 160). If we consider mixtures (i.e., convex combinations of states), then $T^\wedge$ trivially contains mixtures of elements of T. Indeed, with common rich assumptions (e.g., that states are probability measures), one

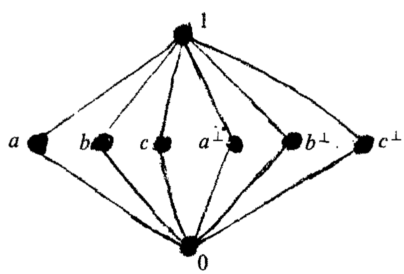

Fig. 2

can characterize the distinction between classical and quantum mechanics in terms of whether all superpositions are mixtures. The reason is essentially that if L is a separable Boolean σ-algebra of subsets of a given set X, then the pure probability measures on L are precisely the so-called concentrated measures (Varadarajan, 1968, Theorem 6.6; Gudder, 1970, p. 1038, Theorem 2). This implies that there are no pure superpositions (i.e., if α and all of T are pure, then $\alpha \in T^\wedge$ iff $\alpha \in T$) and every $^\wedge$-closed set of states is determined by a subset of X. It also implies that L is isomorphic to $\Sigma^\wedge$, so that $\Sigma^\wedge$ is Boolean.

Since in Hilbert space quantum theory there is a corresponding isomorphism, Varadarajan suggested that the superposition principle be defined as asserting that L is isomorphic $\Sigma^\wedge$. Gudder proved this principle for a general quantum logic, assuming the existence of supports, strong ordering, and states as probability measures (Gudder, 1970, pp. 1039–1040, Theorem 4; see also Beltrametti and Cassinelli, 1976, pp. 377–383; 1981, pp. 120–122). Gudder's isomorphism map is

$$\mathrm{S1}: \quad a \in L \to \mathrm{S1}(a) \in \Sigma^\wedge; \quad \text{with inverse } T \to s(T)$$

We shall briefly describe how our weaker assumptions imply the same isomorphism (the pattern of proofs, omitted here, is like Gudder's). There are three reasons for doing so. (1) Section 3.2 raised the question whether every $^\wedge$-closed set is an $\mathrm{S1}(a)$, i.e., a ran(Ga). This isomorphism makes the answer Yes. (2) This isomorphism gives us another proof of the identity of $\Sigma^\wedge$ and $\Sigma\#$. (3) This isomorphism maps the orthocomplement $^\perp$ in L onto the orthocomplement I on $\Sigma^\wedge = \Sigma\#$.

Since $\mathrm{S1}(a)$ is $^\wedge$-closed, the map $\mathrm{S1}: a \in L \to \mathrm{S1}(a)$ is into $\Sigma^\wedge$. Since states are isotone, it is order-preserving. To show that it has an inverse, we use supports of sets of states. We easily have the two lemmas (using strong ordering for the second), and then the superposition principle:

T.4.13. If T is a nonempty subset of Σ, then $T^\wedge = \mathrm{S1}(s(T))$.

T.4.14. For all $a \in L$, $a \neq 0$, $a = s(\mathrm{S1}(a))$.

T.4.15. $a \to \mathrm{S1}(a)$ is an isomorphism from L to $\Sigma^\wedge$ with $T \to s(T)$ as inverse.

Let us state separately the Yes answer to Section 3.2's question:

Corollary to T.4.15. If $T \in \Sigma^\wedge$, then $T = \mathrm{S1}(s(T)) = \mathrm{ran}(G(s(T)))$.

Reasons 2 and 3 above involve our second main concept, inaccessibility, to which we now turn. [For lack of space, we cannot further discuss superposition; e.g., the connection with sectors (Pták and Pulmannová, 1991, pp. 61–78).] We used the idea of disturbing measurement to motivate inaccessibility (cf. Section 4.1). This kind of strategy goes back at least to Pool (1968), who deduced some features of L from axioms governing the state-transition maps Ga, using especially results of Foulis (1960) (see, e.g., Beltrametti and Cassinelli, 1981, pp. 177–190). And Section 4.1's definition of inaccessibility is now standard [e.g., it is the orthogonality of Gudder (1970, p. 1040), Beltrametti and Cassinelli (1981, p. 121), and Pták and Pulmannová (1991, p. 76)]. Of course, T.3.4 assures that any definition of I as symmetric and irreflexive makes the map I an orthocomplement on the lattice $\Sigma\#$ of I-closed sets. But this definition of I meshes with $^\perp$ in the sense of reasons 2 and 3 above. Thus one can prove (again, with only our assumptions about states and L) that:

T.4.16. For all $T \subseteq \Sigma$: $I(T) = \mathrm{S1}(s(T)')$; $T^\wedge \subseteq T\#$; $T\# \subseteq T^\wedge$; and the maps $a \to \mathrm{S1}(a)$ and its inverse $T \to s(T)$ preserve orthocomplements: $\mathrm{S1}(a^\perp) = I(\mathrm{S1}(a))$; and $s(I(S)) = s(S)^\perp$.

The pattern of proof is in the order stated; one can follow, e.g., Beltrametti and Cassinelli (1981, pp. 300–301, C.2.4ff); the third and therefore fourth assertions (but not the first nor second; cf. our own T.4.12!) need strong ordering.

As regards reason 2 above, we should also note what seem to us to be the first results that L's closure maps coincide ($\perp\perp = \mathrm{L0S0}$), and that Σ's do ($II = \mathrm{S0L0}$), where, contrary to our approach, the result does not assume that the other pair of maps coincides. Zierler (1961, p. 157, Lemma 1.14) seems to be the first such "direct" proof that L's closure maps coincide. He uses the terminology of cuts to describe $\perp\perp$, and he assumes that each state has support and that Σ is strongly ordering. On the other hand, Guz (1978, p. 6, Lemma 5) gives a direct proof that Σ's closure maps coincide, starting from similar assumptions.

Finally, we should remark that the assumption of supports is especially interesting in that in the form (2), just before T.4.6, it is the assumption of the Jauch and Piron (1963) no-hidden-variables theorem, criticized by Bell (1966, p. 450) and Bohm and Bub (1966, p. 474) [for discussion, see Jammer (1974, pp. 305–306, 317–318)].

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